Journal of Statistical Physics, Vol. 92, Nos. 5/6, 1998

Nonlinear Wave Propagation in a Disordered Medium

Jared C. Bronski[1]

Received September 23, 1997; final March 19, 1998

In this paper we consider the problem of solitary wave propagation in a weakly disordered potential. Through a series of careful numerical experiments we have observed behavior which is in agreement with the theoretical predictions of Kivshar *et al.*, Bronski, and Garnier. In particular we observe numerically the existence of two regimes of propagation. In the first regime the mass of the solitary wave decays exponentially, while the velocity of the solitary wave approaches a constant. This exponential decay is what one would expect from known results in the theory of localization for the linear Schrödinger equation. In the second regime, where nonlinear effects dominate, we observe the anomalous behavior which was originally predicted by Kivshar *et al.* In this regime the mass of the solitary wave approaches a constant, while the velocity of the solitary wave displays an anomalously slow decay. For sufficiently small velocities (when the theory is no longer valid) we observe phenomena of total reflection and trapping.

KEY WORDS: Nonlinear Schrödinger equation; nonlinear scattering; disordered media.

I. INTRODUCTION

For many years the effect of disorder and the phenomenon of localization has been one of great interest in statistical physics. For many linear equations, such as the Schrödinger equation with a random potential, the effects of the disordered potential are well understood, both at a physical level and a rigorous mathematical level. One natural generalization is to consider the effects of such a disordered potential on a nonlinear equation.[3, 6–8, 15, 14] There are a number of physical situations in which the study of such a nonlinear evolution is natural. For instance if one considers the problem of many interacting particles moving in a common disordered

[1] Department of Mathematics, Stanford University, Stanford, California 94305.

medium, then under the usual mean field approximation (Hartree–Fock approximation) the evolution is governed by a nonlinear Schrödinger type equation. In the case where the particles are in one dimension, and interact via a delta function interaction, the Hartree–Fock equation is exactly the one dimensional cubic nonlinear Schrödinger equation. The cubic NLS also arises in fiber optics, and if we assume that, due to manufacturing variations, the fiber properties (such as group velocity dispersion) vary randomly along the length of the fiber, a random NLS is an extremely interesting equation to study.

The model which we will consider in detail in this paper is that of a single NLS solitary wave propagating through a medium consisting of randomly placed scatterers, which may be modelled by delta functions or some smooth but rapidly decaying function. Such a model was first analyzed by Kivshar, Gredeskul, Sánchez and Vásquez[13] using perturbation theory based on the inverse scattering transform. This analysis was later extended by the author[2] and by Garnier.[9, 10] The basic prediction of these calculations is that, for an NLS solitary wave propagating in a disordered potential there are two distinct regimes of behavior, which depend on the initial value of a, the ratio of the amplitude of the solitary wave to the velocity of the solitary wave.

For sufficiently small α the mass of the solitary wave decays to zero exponentially as the solitary wave propagates into the medium, with the lost mass being scattered into dispersive radiation modes. The velocity of the solitary wave, on the other hand, approaches a constant. For large enough a, however, it is found that the mass of the solitary wave approaches a *constant* after a large number of scattering events, while the *velocity* decays to zero very slowly (like $\ln^{-1}(N)$, where N is the number of scattering events). Further these two asymptotic parameter regimes are consistent, in the following sense—if α is small then a decreases during a scattering event, while α increases during a scattering event if α is initially large. However the theory is not valid for small velocities, when the kinetic energy of the solitary wave is comparable in size to the height of the disordered potential. In this parameter regime there exists the possibility of total reflection of the solitary wave, and trapping phenomena. Since the velocity decays to zero in the strongly nonlinear regime, we expect that such phenomena will eventually be important.

This formal calculation is particularly interesting in light of the conjecture of Frohlich, Spencer and Wayne[8] that solutions to the nonlinear Schrödinger equation localize for sufficiently small initial data. The above formal calculation, if correct, implies that the assumption of small initial data is necessary, and for large data there are states which *do not* localize (at least on the time-scales where the theory is expected to be valid). Thus it is interesting to attempt to verify the formal asymptotics through numerical simulations. In this paper we present detailed numerical calculations which verify this theory. Both of the above mentioned regimes are observed in the numerical experiments, and the observed behaviors are in good qualitative agreement with the predictions of the theory.

II. BACKGROUND

In this section we review the theory of solitary wave propagation in a disordered medium. The particular model of interest, which was first studied by Kivshar *et al.*[13] and later extended by the author[2] and Garnier,[9, 10] is that of a solitary wave or solitary wave propagating in a weak disordered potential made up of many randomly placed "copies" of a basic scatterer $V(x)$,

$$i\psi_t = \psi_{xx} + 2\,|\psi|^2\,\psi + \varepsilon \sum_i \gamma_i V(x - x_i)\,\psi$$

where the locations x_i and the strengths γ_i are random. (In the original works the strengths of the scatterers were fixed. However the extension to the case where the strengths of the scatterers are chosen randomly is relatively straightforward.) The initial condition is taken to be a single incident solitary wave with mass a_I and velocity b_I:

$$\psi(x, 0) = a_I\,\mathrm{sech}(a_I(x - 2b_I t))\,e^{i(b_I x + (b_I^2 - a_I^2)\,t)}$$

which is an exact solution in the case where $\varepsilon = 0$.

This equation has already been partially non-dimensionalized by rescaling x so that the dispersion coefficient (which has units *length*^{2}*/time*) is unity, and rescaling the amplitude so that the coefficient of the nonlinearity is equal to two. It is interesting fully nondimensionalize the equation since the dimensionless parameters associated with the problem provide a great deal of insight into what effects are expected to be important. This non-dimensionalization is most easily accomplished by making the rescaling

$$\tilde{t} = b_I^2 t$$
$$\tilde{x} = b_I x$$
$$a_I \tilde{\psi} = \psi$$

After this resealing the equation becomes

$$i\tilde{\psi}_{\tilde{t}} = \tilde{\psi}_{\tilde{x}\tilde{x}} + 2\,\frac{a_I^2}{b_I^2}\,|\tilde{\psi}|^2\,\tilde{\psi} + \frac{\varepsilon}{b_I^2} \sum \gamma_i V\left(\frac{x}{b_I} - x_i\right)\tilde{\psi}$$

There are three dimensionless parameters. The first is the ratio ε/b_I^2. This can be thought of as the ratio of the height of the background potential to the kinetic energy of the soliton. In this paper this quantity is our perturbation parameter, and is always assumed to be small, so that the kinetic energy of the soliton is much greater than the height of the barrier. Basically this is excluding effects like capture or reflection of the soliton by the background potential.

The second parameter is $\alpha^2 = a_I^2/b_I^2$. This can be thought of as the ratio of the binding energy of the soliton to the kinetic energy of the soliton, and provides a measure of the importance of nonlinearity in the problem—of how tightly bound the soliton is. It is in this parameter that the change in behavior occurs—for a small (a loosely bound soliton) we find an exponential decay of the soliton mass, while for a sufficiently large (a tightly bound soliton) there is asymptotically no decay of the mass.

The third dimensionless parameter is the product of b_I times some measure of the average length scale associated with potential V. A more careful analysis (see previous work by the author[2]) shows that the width of the strip of analyticity is the correct measure of the average length scale of V. This parameter does not play a particularly interesting role in the theory, and for the most part makes only quantitative and not qualitative changes in the results. The exception is when the width of the strip of analyticity diverges, and V has stronger smoothness properties. This will be discussed further in a later section.

If the potential is weak, $\varepsilon \ll b^2$, then interacting with the scatterer the solution will look like a soliton with slightly different mass and velocity, plus some dispersive radiation. Assuming that the case of a single weak scatterer it is possible to use perturbation theory to calculate the amount of mass scattered from the solitary wave into dispersive radiation. This gives an expression for the mass and velocity of the transmitted solitary wave in terms of the mass and velocity of the incident solitary wave:

$$a_T = a_I - \varepsilon^2 h_1(a_I, b_I)$$
$$b_T = b_I - \varepsilon^2 h_2(a_I, b_I)$$

Here a_I and b_I are the asymptotic mass and velocity of the solitary wave before the scattering event, while a_T and b_T are the asymptotic mass and velocity of the solitary wave after the scattering event. The change of mass of the solitary wave is due to the fact that mass is transferred from the solitary wave into dispersive radiation modes. The total mass, solitary wave mass together with radiation mass, is conserved. There is also an $O(\varepsilon)$ change in the solitary wave velocity during the scattering event which decays to zero as the solitary wave moves away from the scatterer and thus does not effect the asymptotic velocity of the solitary wave. This result is somewhat like a "collective coordinate" type approach, but it includes the loss of mass and velocity due to the shedding of dispersive radiation. This kind of radiative damping is quite difficult to include in a collective coordinate type ansatz, and is usually neglected. This is an important distinction, since it is exactly this slow decay of the soliton parameters due to the shedding of dispersive radiation which leads to the existence of the two regimes of propagation.

Under the additional assumption that the scatterers are well-separated, that the average distance between scatterers is much greater than a solitary wave width, the map giving the change in soliton parameters due to a single scatterer can be iterated to give a discrete dynamical system relating the solitary wave mass/velocity after the nth scattering event to the solitary wave mass/velocity after the $(n+1)$th scattering event:

$$a_{n+1} = a_n - \varepsilon^2 h_1(a_n, b_n)$$
$$b_{n+1} = b_n - \varepsilon^2 h_2(a_n, b_n)$$

Again the change in the solitary wave mass and momentum are due to the mass/momentum which is carried by the dispersive radiation field. Taking the obvious continuum limit of this discrete dynamical system leads a set of ordinary differential equations for a, b as a function of the number of scattering events,

$$a_Z = h_1(a, b)$$
$$b_Z = h_2(a, b)$$

where the continuum variable Z is proportional to $\varepsilon^2 N$, with N the number of scattering events.

For the analysis it is somewhat more natural to rewrite this system in terms of b and $\alpha = a/b$. It is found that there are two possible asymptotic behaviors, depending on the size of α. For sufficiently large α and $V(x) = \delta(x)$ we have

$$\alpha_Z = \alpha^3 B(\alpha) \exp\left(-\frac{\pi\alpha}{2}\right)$$

where $B(\alpha)$ is a function which is of sub-exponential growth for large ε. (A similar though more complicated expression exists for a general potential $V(x)$.) For large Z the solution to this differential equation behaves like

$$\alpha(Z) \approx \frac{2}{\pi} \ln(Z) + o(\ln(Z))$$

which implies that the velocity $b(Z)$ decays like

$$b(Z) \approx \frac{c}{\ln(Z)} + o(\ln^{-1}(Z))$$

and the mass $a(Z)$ approaches a constant value. Interestingly the exact form of $B(\alpha)$ does not matter for the leading order asymptotics, as long as $B(\alpha)$ is sub-exponential, though the form of B *does* change the next order corrections; A similar logarithmic behavior holds for any $V(x)$ which is analytic in some strip about the real axis when analytically continued into the complex plane. For potentials with stronger analyticity properties, for instance a $V(x)$ which is entire, α grows more slowly than logarithmically while the velocity decays more slowly than $\ln^{-1}$.

In the opposite regime, when α is sufficiently small, the behavior is very different. In this limit

$$\begin{aligned} &\alpha(Z) \to 0 \qquad \text{exponentially} \\ &a(Z) \to 0 \qquad \text{exponentially} \\ &b(Z) \to \text{constant} \end{aligned}$$

The exponential decay of $\alpha(Z), a(Z)$ is, of course, what one would expect from consideration of the linear problem.

Notice that the large α approximation is consistent in the sense that, if α is initially large then α increases during a scattering event—the problem in some sense becomes more nonlinear after many scattering events. Similarly if α is sufficiently small then α decreases during a scattering event. Note, however, that in the first regime the velocity b is predicted to decay to zero. The perturbation argument used to derive these equations only holds when $\varepsilon/b^2 \ll 1$. Since $b \to 0$ this approximation will eventually break down. When the kinetic energy of the solitary wave is comparable to the height of the potential there arises the possibility of a whole host of new phenomenon, including capture of and total reflection of a solitary wave. We actually observe these phenomena in the numerical simulations.

For more details on the formal derivations of these results we refer the interested to the papers of Kivshar *et al.*,[13] Garnier[10, 9] and the author.[2] The purpose of this paper is to present a series of numerical simulations of the full partial differential equation which are intended to provide evidence in support of the above theory. By doing these computations in a sliding "window" which moves with the solitary wave we are able to simulate many thousands of solitary wave/scatterer collisions in a computationally inexpensive way. This allows us to observe quite clearly the dramatic change in behavior as the ratio of the incident mass to the incident velocity is varied. When the ratio of the incident mass to the incident velocity is small we observe an exponential decay of the mass, as one would expect from consideration of the linear problem. When this ratio is sufficiently large, so that nonlinear effects are important, we find a new regime of propagation, where the mass of the solitary wave approaches a constant and the velocity displays an anomalously slow decay.

III. NUMERICAL EXPERIMENTS

A. The Numerical Method

In this section we present some numerical simulations of the following perturbed NLS equation,

$$i\psi_t = \psi_{xx} + 2\,|\psi|^2\,\psi + \varepsilon \sum_i \gamma_i V(x - x_i)\,\psi$$

where, as in the previous section, the strengths of the scatterers γ_i and the locations of the scatterers x_i were chosen from a uniform random distribution. The locations x_i were distributed on $[0, L]$, and the strengths γ_i were distributed on $[-1, 1]$.

The nonlinear Schrödinger equation, like the Schrödinger equation, is Galilean invariant—if ψ satisfies the NLS equation

$$i\psi_t = \psi_{xx} + |\psi|^2\,\psi$$

then $\tilde{\psi}$ defined by

$$\tilde{\psi} = e^{i(vx + v^2 t)}\psi(x + 2vt, t)$$

else satisfies the NLS equation

$$i\tilde{\psi} = \tilde{\psi}_{xx} + |\tilde{\psi}|^2\,\tilde{\psi}$$

and represents the same solution in a coordinate system moving with velocity $2v$. The numerical experiments were performed in a Galilean reference frame which was moving with and centered on the solitary wave. The instantaneous center of mass and center of mass velocity were calculated at a fixed time using the representations

$$X_{COM} = \frac{\int x\,|\psi|^2\,(x, t)\,dx}{\int |\psi|^2\,(x, t)\,dx}$$

and

$$V_{COM} = \frac{dX_{COM}}{dt} = \frac{\int i\psi_x \bar{\psi} - i\bar{\psi}_x \psi \, dx}{\int |\psi|^2 \, dx}$$

and a Galilean transformation and a translation were performed to move to the coordinate system centered on and moving with the center of mass of the pulse. Unfortunately the center of mass velocity is not exactly constant, due to the presence of the perturbing potential, and thus the soliton will slowly drift away from the center of the computational domain. When the center of mass has drifted a prescribed distance from the center of the computational domain the program calculates a new center of mass and center of mass velocity and performs a Galilean transformation to this new moving frame. This periodic updating of the reference frame would not, of course, be necessary if the perturbations had been absent. However since the change in the velocity of the solitary wave is small, due to the smallness of the perturbation, this updating of the Galilean frame need not be done every time step, and is computationally inexpensive.

In the coordinate system moving with velocity V_{COM}, the instantaneous center of mass velocity, and centered on the solitary wave location the perturbed NLS equation becomes

$$i\psi_t - \psi_{xx} - 2 |\psi|^2 \psi = \varepsilon \sum_i V(x - x_i + V_{COM} t) \psi = \varepsilon V(x, t) \psi$$

In this representation the solitary wave remains fixed, and the perturbing potential is time-dependent. This is a great savings in computational effort, since it becomes possible to work on a computational domain of fixed size, and compute only in this domain. Had we worked in a fixed computational domain we would have had to take a computational domain at least as wide as the total distance travelled by the solitary wave. For long time runs this would be *extremely* computationally expensive—in most runs the full domain is $2^{18} \approx 2.6 \times 10^5$ points. By working in the Galilean reference frame we are able to perform this computation with 256 or 512 points which move along with the solitary wave. This scheme basically the same as the one employed by Knapp [14] in some very interesting work on a similar solitary wave scattering problem. Note, however, that the experiments by Knapp are in a very different parameter regime. In the present work we are primarily interested in the strongly nonlinear regime, which is not addressed in the work of Knapp.

The numerical solutions were calculated using a split-step pseudospectral code. The linear step, satisfying

$$i\psi_t = \psi_{xx}$$

was computed using a standard FFT package, while the nonlinear step, satisfying

$$i\psi_t = 2\,|\psi|^2\,\psi + \varepsilon V(x, t)\,\psi$$

was computed using a fourth order ODE solver. The infinite line boundary conditions were simulated by adding a small imaginary part to $V(x, t)$ supported in a strip of width L_{abs} near the boundaries, to damp outgoing radiation. The absorbing function was taken to be

$$V_{IMAG} = \begin{cases} \cos\left(\dfrac{\pi x}{2L_{abs}}\right) & 0 < x < L_{abs} \\ 0 & L_{abs} < x < L - L_{abs} \\ \cos\left(\dfrac{\pi(L-x)}{2L_{abs}}\right) & L - L_{abs} < x < L \end{cases}$$

In all experiments the width of the computational domain was chosen to be sufficiently large that the damping of the solitary wave can be neglected, and the damping affects only the dispersive radiation.

The scatterers $V(x)$ were, unless otherwise noted, taken to be hyperbolic secants. The width of the scatterers was comparable to but smaller than the width of the incident solitary wave.

In these experiments the computational domain was taken to be $[-30, 30]$. The solitary wave parameters are in the range $a, b \in [0.5, 1.5]$. The computational domain is discretized into 256 or 512 points, so Δx is in the range 0.125 to 0.25. The total time of integration is between 1.5×10^4 and 6×10^4, and the number of time-steps between 1.5×10^6 and 6×10^6, for a Δt of approximately 0.01.

B. The Weakly Nonlinear Regime

This experiment is conducted for a solitary wave with incident α below the critical value of α. The mass of the incident solitary wave was 1.5, as was the incident velocity, giving $\alpha = 1$. In this regime the theory predicts that the mass of the solitary wave should decay to zero exponentially. The velocity, on the other hand, is expected to approach a constant. This is to be expected from consideration of the linear problem—in linear scattering some mass is reflected during scattering from a potential, but the asymptotic velocity (wavenumber) is *not* changed.

The first figure (Fig. 1a, b) shows the perturbing potential for the first series of numerical experiments. The potential, which is supported on

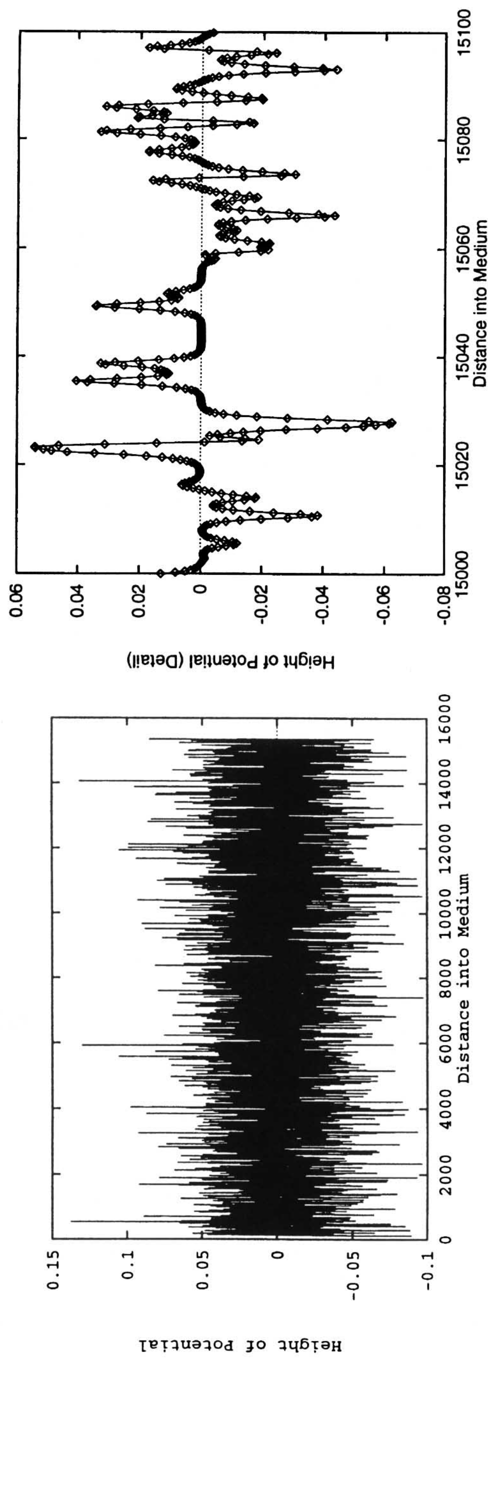

Fig. 1. Background potential—"weakly nonlinear" regime.

[0, 61440], is made up of 1.8×10^5 copies of a single scatterer distributed randomly throughout the medium. This potential is discretized into 262144 points, resulting in $\Delta x \approx 2$. Again the actual computations are carried out on a computational "window" of 256 gridpoints which tracks the center of mass of the solitary wave.

The next graphs (Fig. 2) show the mass and velocity of the solitary wave as a function of time. The computational domain is chosen to be wide enough that the decay of the solitary wave part of the solution due to the absorbing boundaries can be neglected, and thus the decay of the mass represents the mass lost to dispersive radiation. The decay of the solitary wave mass to zero is apparent. The solid line represents a least squares fit to a exponential profile, and shows good agreement to the mass profile for large times. Again this exponential decay of the mass is what one would expect from the linear theory. It is also apparent that the velocity quickly approaches a constant, as predicted by the theory. The "oscillations" in the graph of the velocity are due to the fact that the velocity changes to order ε while it is interacting with the scatterer. However there is no net change in the velocity to order e after the interaction—the velocity of the solitary wave before it interacts with the scatterer is differs from the velocity of the solitary wave after it interacts with the scatterer at $O(\varepsilon^2)$. It is this $O(\varepsilon^2)$ loss in velocity which is responsible for the drift of the mean velocity downwards. This plot is in good agreement with the theory, which predicts that the solitary wave velocity would approach a constant after a large number of scattering events (long time).

This exponential decay of the solitary wave mass seems to provide evidence in support of the conjecture of Frohlich, Spencer and Wayne[8] that solutions of the random nonlinear Schrödinger equation localize for *sufficiently weak* nonlinearity.

The next plot (Fig. 3) is a detail from the graph of solitary wave velocity vs. time in the previous figure (Fig. 2). Here we see the $O(\varepsilon)$ changes in the velocity during the scattering event (the velocity increases for interactions with attractive potentials and decreases during interactions with repulsive potentials. In addition we can also see the slow $O(\varepsilon^2)$ drift of the velocity, as predicted by the theory.

Lastly Fig. 4 shows the alpha parameter, which is the ratio of the mass to the velocity. This parameter is, in a sense, a measure of the nonlinearity of the problem. As in the previous two plots this parameter shows an exponential decay for long times, in agreement with the theory. This illustrates the interesting (and non-obvious) prediction that, if the nonlinearity is initially sufficiently small it becomes *less important* after a large number of scattering events.

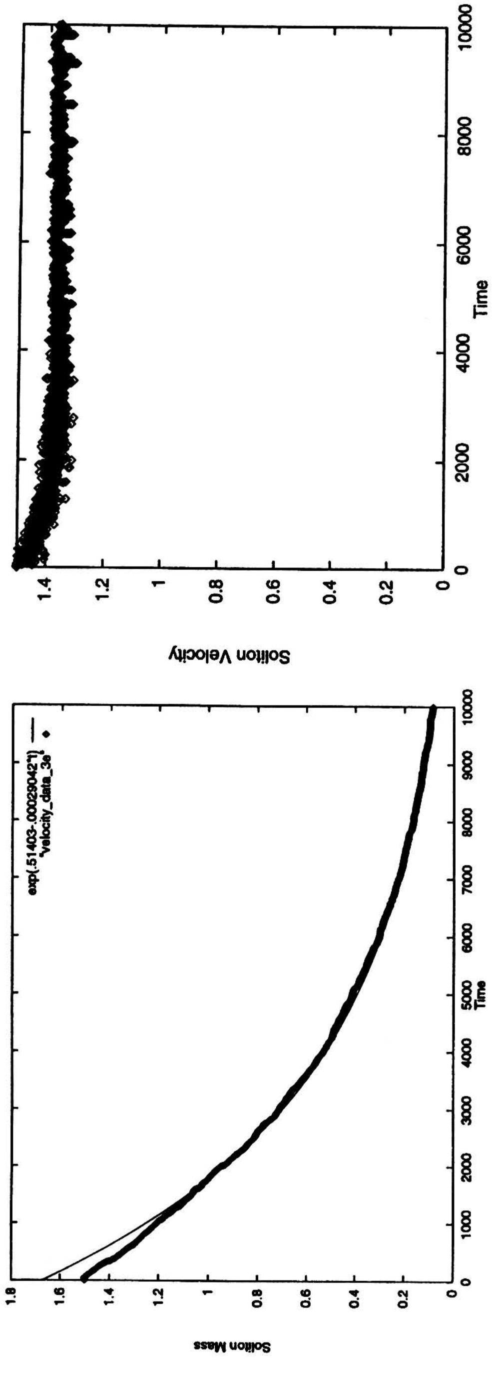

Fig. 2. Solitary wave mass and velocity as a function of time—weakly nonlinear regime.

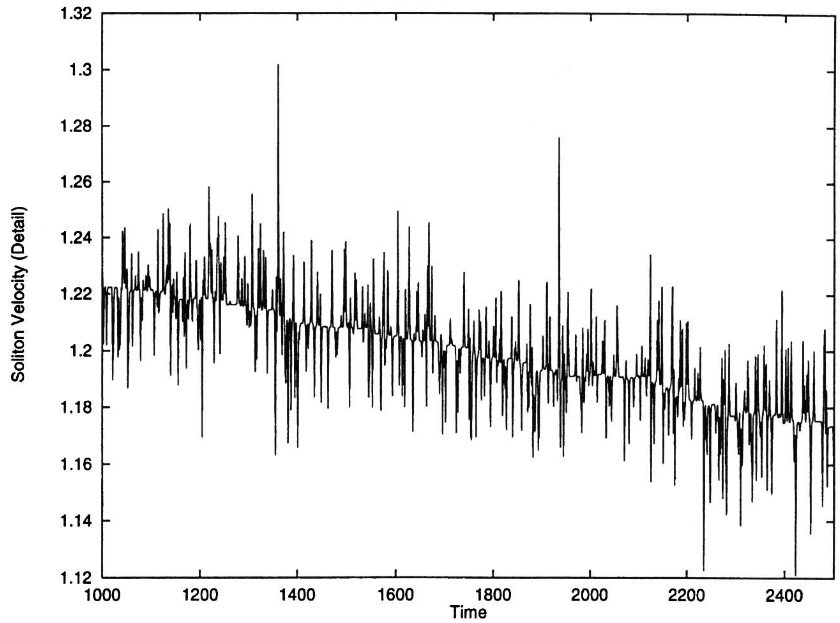

Fig. 3. Detail of plot of velocity—weakly nonlinear regime.

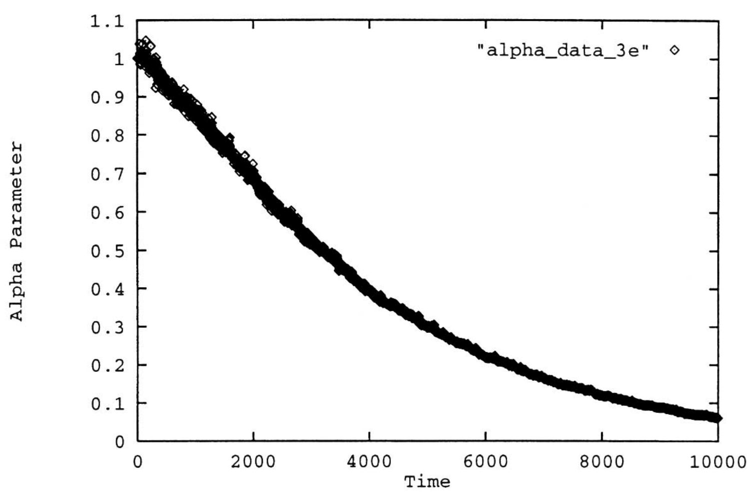

Fig. 4. Alpha parameter as a function of time—weakly nonlinear regime.

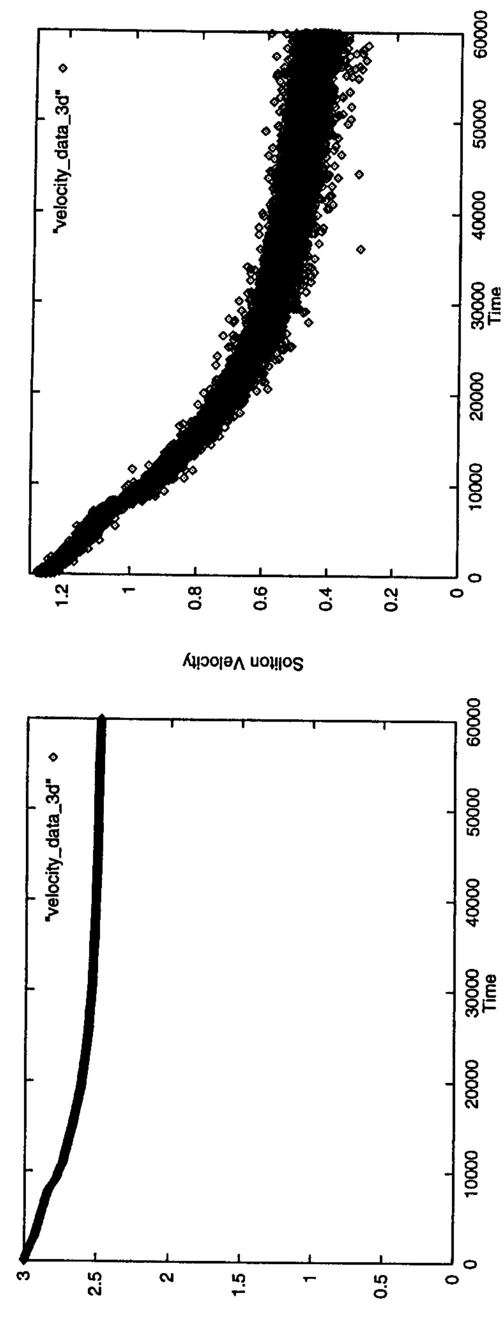

Fig. 5. Solitary wave mass and velocity as a function of time—strongly nonlinear regime.

C. Experiment 2—The Strongly Nonlinear Regime

This experiment is conducted for a solitary wave with incident α above the critical value, so that the solitary wave is in the strongly nonlinear regime. Again the theory predicts that the mass will approach a constant after a large number of scattering events, while the velocity will decay to zero extremely slowly (like $\ln^{-1}(Z)$, where Z is a measure of the number of scattering events.) This behavior is confirmed by the numerical experiments.

This experiment was conducted using a incident solitary wave with an initial mass of 3 and an initial velocity of 1.25, giving an initial α of 2.4. It is apparent from Fig. 5 that the mass is approaching a constant value, in accordance with the theory, and the velocity seems to be decaying slowly, again in accordance with the theory. Of course it would extremely difficult to numerically simulate a sufficient number of scattering events to verify $\ln^{-1}(Z)$ decay, but Fig. 5 displays the same qualitative behavior predicted by the theory. Finally Fig. 6 shows the behavior of the α parameter, which is the ratio of the solitary wave mass to the solitary wave velocity. This quantity is expected to *grow* with the number of scattering events—the velocity is decreasing more rapidly than the mass. It is easily seen from Fig. 6 that the numerical experiments support this conclusion.

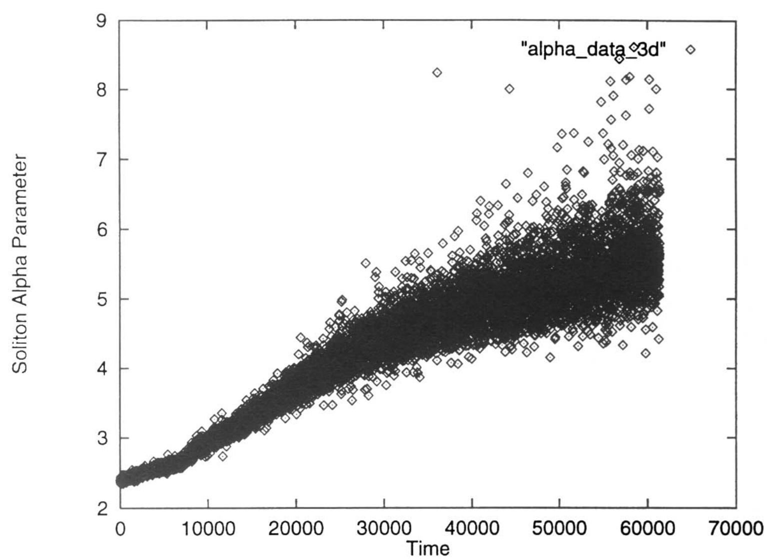

Fig. 6. Alpha parameter as a function of time—strongly nonlinear regime.

D. The Critical Value α^*

The previous two experiments show quite clearly the rather sharp change in the propagation properties of the solitary wave as the α parameter is changed from $\alpha = 1$ to $\alpha = 2.4$. One of the predictions of the Kivshar theory is that α, the ratio of solitary wave amplitude to solitary wave velocity, should *decrease* after a scattering event for small α and *increase* during a scattering event for large α. This implies that there should be a critical value of α, denoted by α^*, at which a stays the same during a scattering event. This critical α is where this change in behavior occurs. In the case of delta function scatterers the paper of Kivshar gives a transcendental equation for α^*, which can be evaluated numerically to give $\alpha^* \approx 1.25$. In this experiment we attempt to determine this critical value α^* via direct numerical solution of the partial differential equation. Figure 7 shows graphs of α vs. time for solitary waves with different initial values of α. It is clear from this graph that for small initial values ($\alpha = 1.0, 1.25$) α decreases, for large initial values ($\alpha = 1.75, 2.0$) α increases, and the stationary point occurs at approximately $\alpha \approx 1.5$. This value is in qualitative agreement with the prediction of Kivshar *et al.* for the critical value. Of course these numerical experiments were not conducted using delta function scatterers, but rather with scatterers of some finite width (in this case hyperbolic secants). Simple heuristic arguments suggest that smoother scatterers

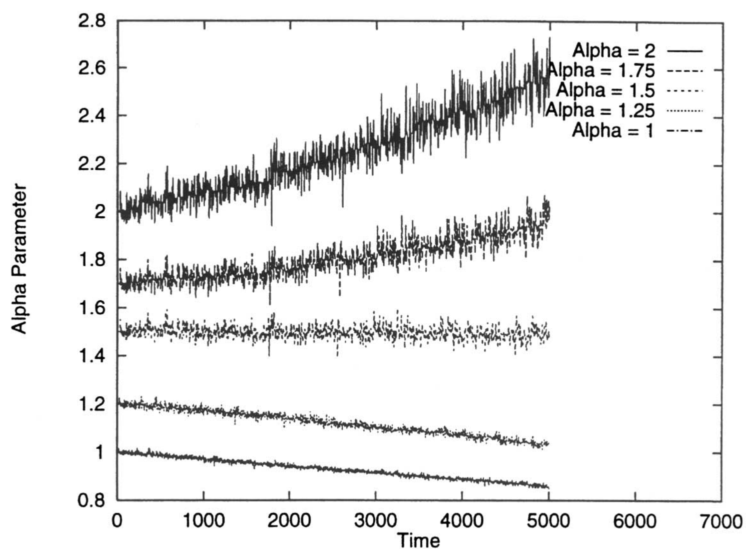

Fig. 7. Alpha vs. time for different initial values.

should *increase* the value of α^*, so we expect that the main discrepancy between the predicted value and the numerically determined value is due to the finite width of the scatterers.

E. Scattering and Analyticity Properties

In previous work[2] it was shown that the analyticity properties of the scatterer play an important role in the case of strongly nonlinear scattering. In particular it was shown that the amount of dispersive radiation excited should be significantly less for (four example) scatterers which are entire functions (when considered as functions of a complex variable) rather than meromorphic functions. In this experiment we attempt to verify this conclusion numerically.

For this experiment we compare the scattering of an incident solitary wave by two different potentials. The first is, as in previous experiments, made up of a number of randomly placed hyperbolic secant scatterers:

$$V(x) = \sum_i \gamma_i \operatorname{sech}(x - x_i)$$

As before γ_i and x_i are uniformly distributed random variables. This potential inherits the analyticity properties of the hyperbolic secant, and is analytic in a strip of width $\pi/2$. We then generate a second potential

$$\tilde{V}(x) = \sum_i \gamma_i \eta \exp(-\beta(x - x_i)^2)$$

Here the γ_i and x_i are chosen to be the same realization as in the hyperbolic secant potential. The constants η and β were chosen to make the Gaussian have the same width as the hyperbolic secant-

$$\int \operatorname{sech}(x)\, dx = \int \eta \exp(-\beta x^2)\, dx$$

$$\int x^2 \operatorname{sech}(x)\, dx = \int \eta x^2 \exp(-\beta x^2)\, dx$$

The potential $\tilde{V}(x)$ is, of course, entire and has no poles in the complex plane.

It is clear from Fig. 8 (which shows a short segment of the disordered potentials) that the potentials $V(x)$ (shown in the solid line) and $\tilde{V}(x)$ (in points) are quite similar. However the amount of radiation generated by $\tilde{V}(x)$, the entire function, is marked less than the amount of radiation

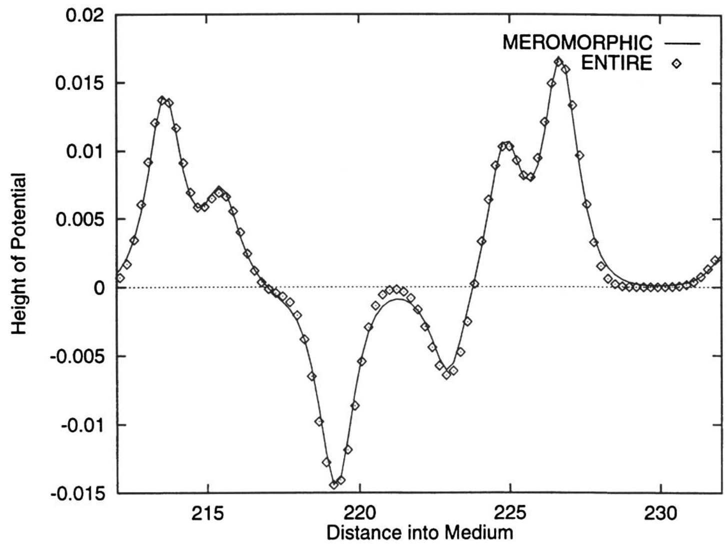

Fig. 8. Comparison—entire and meromorphic potentials.

generated by the meromorphic potential $V(x)$. Figure 9 shows a graph comparing the amount of mass in the solitary wave as a function of time. The solitary wave which is moving through the entire potential loses mass much more slowly than the solitary wave which is moving through the meromorphic potential. Of course this type of behavior is well known in

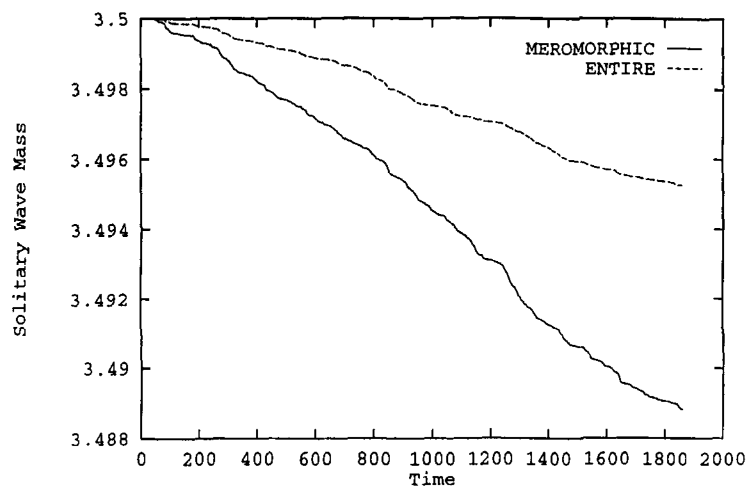

Fig. 9. Solitary wave mass—entire and meromorphic scatterers.

linear scattering theory, where things like the large wavenumber behavior of (for instance) reflection and transmission coefficients sensitive to the analyticity properties of the scattering potential. While similar in spirit to these linear results, the nonlinear result is somewhat different. It arises, for example, for *large amplitude* solitary waves at *fixed wavenumber* (velocity). The typical linear result is for *large wavenumber*, and of course does not depend on the amplitude.

F. Large Time Behavior and Solitary Wave Capture

The theory outlined previously in the paper is valid for weak potentials—it is expected to be valid when $\varepsilon \ll b^2$, the kinetic energy of the incident solitary wave is much less than the height of the potential barrier. This theory predicts that, in the strongly nonlinear regime, the velocity of the solitary wave should decay slowly. When the solitary wave velocity has decayed sufficiently so that it is of comparable size to the background potential ($\varepsilon \approx b^2$) the theory outlined in the previous section is no longer valid, since the effects of the nonlinearity and the effects of the background potential are of comparable size. In this situation one expects to observe different phenomena, including the capture of or total reflection of the solitary wave by the scatterers. This is illustrated in Fig. 10 which depicts the trapping of a solitary wave via repeated reflections from a pair of

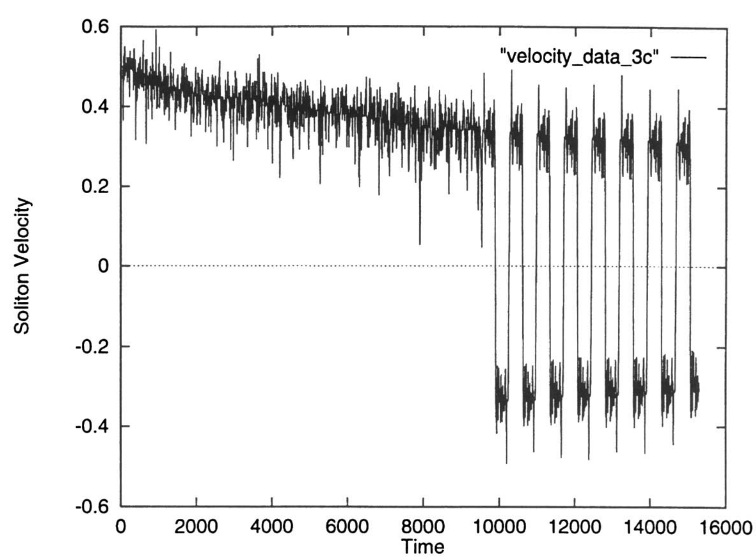

Fig. 10. Velocity as a function of time—solitary wave capture.

relatively large scatterers. The solitary wave propagates and decelerates until it is totally reflected by a scatterer (at a time slightly before $t \approx 10^5$). The solitary wave then propagates backwards through the medium, still losing velocity, until it is reflected a second time by another scatterer, and begins travelling forward again. This process is expected to continue until the solitary wave has radiated all of its kinetic energy away and becomes trapped in some local minima of the background potential.

IV. CONCLUSIONS

In this work we have made a careful numerical study of solitary wave scattering by a disordered collection of weak scatterers. By working in a reference frame which is centered on and moving with the solitary wave we are able to simulate a large number of scattering events efficiently, which allows us to verify the main prediction of the existing theory—the existence of two parameter regimes. The first, which is valid for solitary waves whose binding energy is much smaller than their kinetic energy, consists of nearly linear behavior, with the mass of the solitary wave demonstrating exponential decay, and the velocity approaching a constant value. The second regime is a strongly nonlinear regime where the velocity of the solitary wave shows an anomalously slow decay, while the mass of the solitary wave approaches a constant. When the velocity has decayed sufficiently the scatterers can no longer be considered weak and other phenomena such as capture or total reflection of the solitary wave are observed. It is worth noting that such phenomena are expected to be well described by the widely known collective coordinate theory.

Since the solitary wave appears to always be eventually captured this nonlinear regime could be said to exhibit a form of localization. however because of the anomalously slow decay of the velocity (like $\ln^{-1}$) the localization length can be quite large, and is exponentially large in the amplitude for solitary waves of large initial amplitude. Similar suppression of localization by nonlinearity has been observed by Devillard and Souillard[6] in a time-independent NLS equation.

The work could easily be extended to other models. One model which would be interesting to consider is that of a disordered KdV equation. In the recent experiments of Hopkins, Keat, Meegan, Zhang and Maynard[11] on localization of third sound waves in superfluid helium they observe in laboratory experiments a marked suppression of localization by nonlinear effects. When the third sound waves are of small amplitude the decay of the third sound is exponential with distance into the medium. However at large amplitude the decay is found to be sub-exponential. Since third sound waves in superfluid helium are commonly modelled by the KdV equation[5, 12, 1] the understanding of solitary wave propagation in a disordered KdV would provide a great deal of insight into these experiments. The author and Garnier are currently considering this problem.

The theory which we have attempted to verify here is, of course, still at the level of a formal calculation and has not been made rigorous. There is some hope, however, that the present formal calculation could be rigorously justified. In recent work Buslaev and Perelman[4] and Softer and Weinstein[16] have rigorously treated the radiative damping of bound states in nonlinear Hamiltonian wave equations which are perturbations of linear wave equations. It is possible[17] that these techniques could be applied to the problem outlined here, with the linearized NLS operator playing the role of the underlying linear wave equation. The recent work of Garnier[10] has also made significant progress towards making this theory rigorous.

REFERENCES

1. A. C. Biswas and C. S. Warke, Nonlinear effects in two-dimensional superfluid ^{4}He, *Phys. Rev. B* **22**(5):2581–2584 (1980).
2. J. C. Bronski, Nonlinear scattering and analyticity properties of solitons, *J. Nonlin. Sci.* **8** (1998).
3. J. C. Bronski, D. W. McLaughlin, and M. J. Shelley, Stability of time-harmonic localized solutions to a random NLS equation, *J. Stat. Phys.* **88**(5/6) (1997).
4. V. S. Buslaev and G. S. Perel'man, On the stability of solitary waves for nonlinear Schrödinger equations, *AMS Translations* **164**:75–98 (1995).
5. C. A. Condat and R. A. Guyer, Korteweg–De Vries solitons and helium films, *Phys. Rev. B* **25**(5):3117–3122 (1982).
6. P. Devillard and B. Souillard, Polynomially decaying transmission for the nonlinear Schrödinger equation in a random medium, *J. Stat. Phys.* **43**:423–439 (1986).
7. B. Doucot and R. Rammal, On Anderson localization in a nonlinear random medium, *Europhys. Lett.* **3**:969 (1987).
8. J. Frohlich, T. Spencer, and C. E. Wayne, Localization in disordered, nonlinear dynamical systems, *J. Stat. Phys.* **42**:247 (1986).
9. J. Garnier, Solitons in milieux aléatoires, Ph.D. thesis, Ecole Polytechnique, 1996.
10. J. Garnier, Transmission of solitons through random media, *SIAM J. Appl. Math.* (1998), in press.
11. V. A. Hopkins, J. Keat, G. D. Meegan, T. Zhang, and J. D. Maynard, Observation of the predicted behavior of nonlinear pulse propagation in disordered media, *Phys. Rev. Lett.* **76**(7):1102–1105 (1996).
12. B. A. Huberman, Superfluid solitons in helium films, *Phys. Rev. Lett.* **41**(20):1389–1393 (1978).
13. Yu. S. Kivshar, S. A. Gredeskul, A. Sánchez, and L. Vázquez, Localization decay induced by strong nonlinearity in disordered systems, *Phys. Rev. Lett.* **64**(15):1693 (1990).
14. R. Knapp, Transmission of solitons through random media, *Physica D* **85**:496–508 (1995).
15. R. Knapp, G. Pananicolaou, and B. White, Transmission of waves by a nonlinear medium, *J. Stat. Phys.* **63**(4):567–584 (1991).
16. A. Soffer and M. I. Weinstein, Resonances, radiation damping and instability in hamiltonian nonlinear wave equations. Preprint, 1997.
17. M. I. Weinstein, personal communication.